Cairo University	Cairo University – Institute of Statistical Studies & Research	
	Department of Mathematical Statistics	
	Academic Year: 2016-2017 Final Exam.	
	Date 15-1-2017 Master Level: First Term	

Course Title:	Course Code:	Time:	Exam. Marks:	# Exam. Sheets:
Advanced Probability	MS 607	3 Hours	75	2

Exam. Instructions :

Question (1) (15 Marks)

Let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the order statistics of a random sample of size n from exponential distribution with the probability density function given by

$$f(x) = \frac{1}{\alpha} \exp \frac{-(x-\beta)}{\alpha} \qquad x \geq \beta,$$

where α and β are positive parameters.

a. Obtain the expected value for $Z = \min(X_1, \ldots X_n)$.

b. Obtain the distribution of the sample range $R = X_{(n)} - X_{(1)}$.

c. Let $\overline{X}$ denote the mean of random sample of size 128 from a gamma distribution with $\alpha = 2$ and $\beta = 4$. Compute an approximate value of $P(7 < \overline{X} < 9)$.

Question (2) (14 Marks)

a. Let $\{X_k\}, k = 1,2,3,\ldots$ be sequence of independent random variable defined as

$$P(X_k = -k^{\frac{1}{3}}) = \frac{1}{2} \qquad \text{and,} \qquad P(X_k = k^{\frac{1}{3}}) = \frac{1}{2}$$

Check whether the law of large numbers holds for this sequence.

b. Let U_n denote the nth order statistics of a random sample of size n from a uniform distribution on the interval $(0, \theta)$. Prove that $Z_n = \sqrt{U_n}$ converges stochastically to $\sqrt{\theta}$

c. Let $X_1, X_2, \ldots, X_n$ are independent Poisson with rate 0.5 and for, $n = 25$ approximate the value $P(10 \leq \sum_{i=1}^{n} x_i \leq 28)$.

Question (3) (23 Marks)

a. Show that if $X_1, X_2, ..., X_n$ are independent random variables and each has the uniform distribution with domain (0,1), show that $Z = -\lambda \sum_{i=1}^{n} \ln(x_i)$ has the gamma distribution , where λ is an unknown parameter.

b. If the probability density function of X is given by

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \qquad -\infty < x < \infty$$

Find the probability density function of $Y = e^{-x}$.

c. Find the measure of kurtosis of the distribution that has the characteristic function of the form $\varphi_x(t) = \exp(\frac{-t^2}{5})$.

d. Let $X_1, X_2, ..., X_n$ be identically independent random sample from a Poisson distribution with mean μ. Write $Y_n = \overline{X}_n$, Does $Y_n \xrightarrow{P} \mu$

Question (4) (23 Marks)

a. Prove that the t distribution tends to standard normal distribution as $n \to \infty$.

b. Obtain the mean and variance of a random variable X has the non-central chi-square with the following probability density function

$$f(x) = \frac{1}{2^{\frac{n}{2}}} e^{\frac{-(\lambda+x)}{2}} \sum_{j=0}^{\infty} \frac{\lambda^j x^{\frac{n}{2}+j-1}}{j!\, 2^{2j}\, \Gamma\left(\frac{n}{2}+j\right)} \qquad x > 0,$$

where λ is the non-centrality parameter and n is the degree of freedom.

c. Let $X_1, X_2, ..., X_n$ be a random sample from the distribution with density function $f(x) = xe^{-x}, x > 0$. Find c if it is known that $P(\overline{X}_n > c) = 0.75$ for n=250.

d. Let $X_1, X_2, ..., X_n$ be a random sample from distribution with cumulative distribution function given by

$$F(x) = 1 - x^{-2}, \qquad 1 \le x < \infty \text{ and 0 otherwise.}$$

Find the limiting distribution of (i) $X_{n:n}/\sqrt{n}$ (ii) $(X_{1:n})^n$